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Wavelets and discrete ordinates method in solving one-dimensional nongray radiation problem

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Abstract

The wavelet expansion is used to evaluate the spectral dependence in the solution of the radiative transfer equation (RTE). The radiative intensity is expanded in terms of Daubechies wrapped wavelet functions in the wavelength domain. The discrete ordinates method (DOM) is applied to solve the resulting equations. As an example, the nongray radiation through an absorbing, emitting and nonscattering medium between two infinite parallel plates is analyzed. The plates are assumed to be black at constant temperatures. The RTE is solved in an absorption band. The results for the case of radiative equilibrium are given and compared with those obtained by other methods. The accuracy of the method is thus validated. © 1998 Elsevier Science Ltd. All rights reserved.

Nomenclature

a_λ spectral absorption coefficient
c_i wavelet expansion coefficients of radiative intensity
d spectral line spacing
D distance between two parallel plates
E emissive power σT^4
h_k coefficients in scaling functions
I radiative intensity
M number of discrete ordinates in DOM
N number of wavelet expansion terms of radiative intensity
q_r radiative heat flux
S mean line intensity in equation (15)
T temperature
W wrapped Daubechies wavelet function
x coordinate along medium layer.

Greek symbols
γ line half width in equation (15)
δ Dirac δ-function
ε_λ spectral emissivity
θ zenith angle from x direction
λ wavelength
μ $\cos\theta$
ρ density of absorbing medium
σ Stefan–Boltzmann constant
φ scaling function
ψ wavelet function
ω solid angle.

Subscripts
1, 2 plate 1 or 2
b blackbody
m discrete ordinate directions
r radiative
λ spectrally dependent.

Superscript
* dimensionless quantity.

1. Introduction

The wide application of the radiation phenomena in various industries has provided the impetus for this research. The governing equation for describing the radiation field in an absorbing, emitting and scattering medium is the radiative transfer equation (RTE), which is an integro-differential equation. The radiative properties of the medium needed in solving this equation are usually

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strongly dependent on both temperature and wavelength. Thus it is difficult to obtain either exact or numerical solutions to this equation. In the past few decades, a variety of methods were developed to solve gray medium problems, including exact [1], P-N approximation [2, 3], discrete ordinates method (DOM) [4, 5], etc. To incorporate nongray properties, which are significant only in certain narrow wavelength bands, several approximate band models were developed. Cess et al. [6] and Chan and Tien [7] applied the isothermal-band and nonisothermal-band absorptance in their analysis, respectively. However, these treatments are limited to the systems with small temperature variations. Modest [8] demonstrated that the weighted-sum-of-gray-gases model, where the nongray gas is replaced by a number of gray gases, can be applied to any solution method for the RTE. But the application is limited to nonscattering media and black-walled enclosures. Another widely used model is the picket fence model, where the band radiative properties are approximated by box model. Yucel and Bayazitoglu [9] applied this model in P-N approximation to solve nongray problems. As we know, for most gases and semi-transparent solid, the absorption coefficients change abruptly in the band, so the box model may not be accurate enough. Besides these, Hottela and Sarofilm [10] extended the zonal method to deal with the nongray media, Kim et al [11] solved the spectral averaged RTE using DOM for cases with known temperature profiles.

In the past decade, a new kind of function, wavelet function, has been developed and widely used in signal and numerical analysis. Its advantages in representing rapidly varying functions made it a good tool to approximate the radiative properties and thus attracted the attention of radiation researchers.

Daubechies wavelet is a set of orthogonal, compactly supported functions which satisfy the following equations [12]:

scaling function

$$\varphi(\lambda) = \sqrt{2}\sum_{k=0}^{2n-1} h_k \varphi(2\lambda - k) \tag{1}$$

wavelet function

$$\psi(\lambda) = \sqrt{2}\sum_{k=0}^{2n-1} (-1)^k h_{2n-1-k} \varphi(2\lambda - k) \tag{2}$$

where h_k are the wavelet filter coefficients discovered by Daubechies [12]. The support of the functions is [0, $2n$], which means the functions are zero out of the interval. With these two functions, the wavelets system can be constructed by dilation and translation as follows:

$$\psi_{j,k}(\lambda) = 2^{j/2} \cdot \psi(2^j\lambda - k) \tag{3}$$

which consist of an orthonormal system

$$\int \psi_{j,k} \cdot \psi_{j',k'} \cdot \mathrm{d}\lambda = \delta_{jj'} \cdot \delta_{kk'} \tag{4}$$

where δ is Dirac δ-function.

Newland [13] gave a wavelet expansion of any function $f(\lambda)$ in L^2 space (the square-integrable function space),

$$f(\lambda) = a_0 + \sum_j \sum_k a_{2^j+k} \cdot W(2^j\lambda - k) \quad 0 \leqslant \lambda < 1 \tag{5}$$

where $W(2^j\lambda - k)$ are wrapped Daubechies wavelet functions, which means the regular Daubechies wavelet functions $\psi(2^j\lambda - k)$ are wrapped around the interval $0 \leqslant \lambda < 1$ as many times as necessary to ensure that their whole length is included in the unit interval. The integer j describes the different levels of wavelets, starting with $j = 0$; integer k covers the number wavelets in each level, from $k = 0$ to $2^j - 1$. An efficient algorithm was given to perform the discrete wavelet transform which are to obtain the expansion coefficients a. Then the wavelet functions can be calculated numerically with the inverse procedure.

Since the radiative properties of the nongray media usually vary only in some small bands, the compactly supported wavelets are expected to give better approximations. Bayazitoglu and Wang [14] applied the wavelet expansion and the P-1 approximation to study the radiation in the nongray media bounded by two infinite parallel plates. Bayazitoglu et al. [15] combined the wavelet method and DOM to solve the same problem. It is believed that the wavelet method can be combined with any existing methods developed for gray media to solve nongray problems.

In this paper, the similar procedure as in [15] is applied to solve the RTE only in one absorption band, not in the whole radiation spectral domain. Thus much computer memory and CPU time are saved.

2. Numerical analysis

As shown in Fig. 1, we consider the radiation between two infinite parallel plates which bound an absorbing, emitting and nonscattering medium. The plates are taken to be black at two different fixed temperatures, T_1 (at $x = 0$) and T_2 (at $x = D$). The system is in radiative equilibrium [15]. The governing equation is

$$\mu \frac{\mathrm{d}I_\lambda(\lambda,x,\mu)}{\mathrm{d}x} + a_\lambda(\lambda,x) \cdot I_\lambda(\lambda,x,\mu) = a_\lambda(\lambda,x) \cdot I_{\lambda\mathrm{b}}(\lambda,x) \tag{6}$$

where $\mu = \cos\theta$, and θ is the zenith angle from the x direction [15].

The nondimensional equation of the same form can be obtained by introducing the following parameters

$$x^* = x/D$$
$$a_\lambda^* = a_\lambda \cdot D$$
$$\lambda^* = (\lambda - \lambda_0)/\Delta\lambda$$
$$I_\lambda^* = I_\lambda \cdot \Delta\lambda/(\sigma T_2^4) \tag{7}$$

where D is the distance between the two plates, λ_0 is the

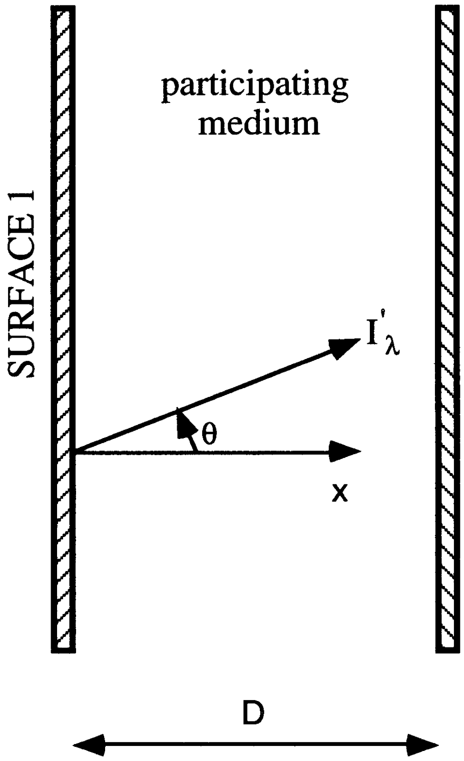

Fig. 1. System geometry.

starting wavelength and $\Delta\lambda$ is the width of the absorption band. Thus $x^* \in [0,1]$ and $\lambda^* \in [0,1]$. The * will be omitted in later discussions.

The energy equation is

$$\nabla \cdot q_r = \mathrm{d}q_r/\mathrm{d}x = 0 \tag{8}$$

which should be solved simultaneously with equation (6).

To solve the nongray RTE (6), the radiative intensity I_λ is approximated by the wavelet expansion in the wavelength domain [15]

$$I_\lambda(\lambda,x,\mu) = a_0(x,\mu) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} a_{2^j+k}(x,\mu) \cdot W(2^j\lambda - k) \tag{9}$$

where $W(2^j\lambda - k)$ are wrapped Dubechies wavelets, J is the maximum level of wavelets, and $2^j + k$ goes from 1 to $N = 2^{J+1} - 1$.

For simplicity, the above expansion can be expressed as

$$I_\lambda(\lambda,x,\mu) = \sum_{i=1}^{N} c_i(x,\mu) \cdot W_i \tag{10}$$

with

$$W_1 = 1 \tag{10a}$$

$$W_i = 2^{j/2} \cdot W(2^j\lambda - k) \quad i = 2,3,\ldots,N \tag{10b}$$

where wavelets W_i ($i = 1,2,\ldots,N$), some of which are shown in Fig. 2, consist of an orthonormal system, which means

$$\int_0^1 W_i \cdot W_{i'} \cdot \mathrm{d}\lambda = \delta_{ii'} \tag{11}$$

where δ is Dirac δ-function.

By substituting expansion (10) into equation (6), applying the Galerkin method, and considering the orthonormal property (11) of the wavelet functions, a set of equations about the expansion coefficients, $c_i(x,\mu)$, are created [15]:

$$\mu \frac{\mathrm{d}c_j(x,\mu)}{\mathrm{d}x} + \sum_{i=1}^{N} c_i(x,\mu) \cdot \int_0^1 a_\lambda(\lambda,x) \cdot W_i \cdot W_j \cdot \mathrm{d}\lambda = \int_0^1 a_\lambda(\lambda,x) \cdot I_{\lambda b}(\lambda,x) \cdot W_j \cdot \mathrm{d}\lambda \tag{12}$$

where $j = 1,2,\ldots,N$.

In matrix form, it becomes

$$\mu \frac{\mathrm{d}C}{\mathrm{d}x} + A \cdot C = B \tag{13}$$

where

$$C = [c_1(x,\mu), c_2(x,\mu), \ldots, c_N(x,\mu)]^T \tag{13a}$$

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \left[\frac{\mathrm{d}c_1(x,\mu)}{\mathrm{d}x}, \frac{\mathrm{d}c_2(x,\mu)}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}c_N(x,\mu)}{\mathrm{d}x}\right]^T \tag{13b}$$

$$A_{i,j} = \int_0^1 a_\lambda(\lambda,x) \cdot W_i \cdot W_j \cdot \mathrm{d}\lambda, \quad i,j = 1,2,\ldots,N \tag{13c}$$

$$B = \left[\int_0^1 a_\lambda(\lambda,x) \cdot I_{\lambda b}(\lambda,x) \cdot W_1 \cdot \mathrm{d}\lambda, \ldots, \int_0^1 a_\lambda(\lambda,x) \cdot I_{\lambda b}(\lambda,x) \cdot W_N \cdot \mathrm{d}\lambda\right]^T \tag{13d}$$

The unknown vector C here is not dependent on wavelength. So it can be solved with the numerical methods developed for gray problems.

In this work, the DOM was used to solve this solid angle dependent equation. It was replaced by a set of equations for a finite number of directions, and the integration over the solid angle was replaced by numerical quadrature. The selection of the ordinate direction μ_m and its quadrature weight w_m depends on the quadrature scheme for approximating the integral. In this work we chose the scheme with equal weights for all directions given by Fiveland [4].

Applied DOM, equation (13) becomes

$$\mu_m \frac{\mathrm{d}C_m}{\mathrm{d}x} + A \cdot C_m = B \tag{14}$$

where $C_m = [c_1(x,\mu_m), c_2(x,\mu_m), \ldots, c_N(x,\mu_m)]^T$, $m = 1,2,\ldots,M$, A and B are as above in (13c, d). This is a set of ordinary

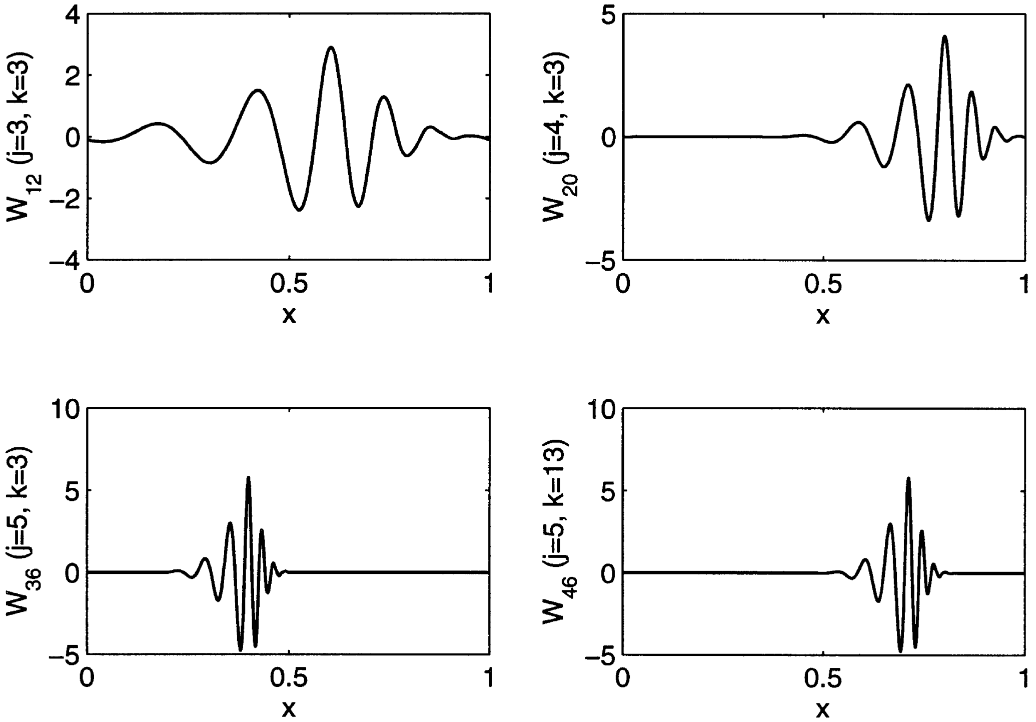

Fig. 2. Some wrapped daubechies wavelet functions.

differential equations about coefficients $c_i(x,\mu_m)$ and can be solved with a Runge–Kutta or a finite difference method when the boundary conditions are given.

Boundary conditions for DOM are generated by expressing the intensity leaving the surface along ordinate direction m as the sum of emitted and reflected intensities. At boundary $x = 0$, it is

$$I_{\lambda m}(x=0) = \varepsilon_{\lambda 1} \cdot I_{\lambda \mathrm{b}1} + 2(1-\varepsilon_{\lambda 1}) \sum_{m'=1}^{M} \mu_{m'} \cdot w_{m'} \cdot I_{\lambda m'} \quad \mu_m > 0 \quad (15)$$

where the summation extends only over the directions with $\mu_{m'} < 0$. Similarly, at boundary $x = D$,

$$I_{\lambda m}(x=D) = \varepsilon_{\lambda 2} \cdot I_{\lambda \mathrm{b}2} + 2(1-\varepsilon_{\lambda 2}) \sum_{m'=1}^{M} \mu_{m'} \cdot w_{m'} \cdot I_{\lambda m'} \quad \mu_m < 0 \quad (16)$$

The summation extends only over the directions with $\mu_{m'} > 0$. Considering that the two infinite parallel plates are black, $\varepsilon_{\lambda 1} = \varepsilon_{\lambda 2} = 1$, the boundary conditions are simplified as

$$I_{\lambda m}(x=0) = I_{\lambda \mathrm{b}1} \quad \mu_m > 0 \quad (17)$$

$$I_{\lambda m}(x=D) = I_{\lambda \mathrm{b}2} \quad \mu_m < 0 \quad (18)$$

Substitute the expansion (10) into the above boundary conditions, multiply the wavelet bases on both sides and integrate over the wavelength domain, with the consideration of the orthonormal property of wavelet functions, we obtain a set of boundary conditions for equation (14) as follows:

$$c_j(x=0,\mu_m) = \int_0^1 I_{\lambda \mathrm{b}1} \cdot W_j \cdot \mathrm{d}\lambda \quad \mu_m > 0 \quad (19)$$

$$c_j(x=D,\mu_m) = \int_0^1 I_{\lambda \mathrm{b}2} \cdot W_j \cdot \mathrm{d}\lambda \quad \mu_m < 0 \quad (20)$$

where $j = 1, 2, \ldots, N$.

Now the expansion coefficients in a particular direction, $c_i(x,\mu_m)$, can be solved from equation (14) when the temperature distribution is given (since A and B in equation (13c, d) are functions of temperature). Then the intensity and heat flux in the direction m can be found by

$$I_m = \int_0^1 I_{\lambda m}\,\mathrm{d}\lambda = c_1(x,\mu_m) \quad (21)$$

$$q_\mathrm{r} = \sum_{m=1}^{M} 2\pi \cdot \mu_m \cdot w_m \cdot I_m = 2\pi \sum_{m=1}^{M} \mu_m \cdot w_m \cdot c_1(x,\mu_m) \quad (22)$$

These are values inside the absorption band. Outside the band, the intensity and heat flux will not change throughout the medium since no absorption occurs there. Then the energy equation becomes

$$\frac{\mathrm{d}q_r}{\mathrm{d}x} = 2\pi \sum_{m=1}^{M} w_m \cdot \mu_m \cdot c_1(x,\mu_m)$$

$$= 2\pi \sum_{m=1}^{M} w_m [B_1 - \sum_{i=1}^{N} A_{1i} \cdot c_i(x,\mu_m)] = 0 \quad (23)$$

To solve the coupled equations (14) and (23), the modified quasilineraization algorithm (MQA) [16] was employed. Defining $f = \mathrm{d}q_r/\mathrm{d}x$, to reach $f = 0$, the temperature adjustment should be

$$\Delta T = -\alpha \frac{f}{\partial f/\partial T} \quad (24)$$

where

$$\frac{\partial f}{\partial T} = 2\pi \sum_{m=1}^{M} w_m \left[\frac{\mathrm{d}B_1}{\mathrm{d}T} - \sum_{i=1}^{N} \frac{\mathrm{d}A_{1i}}{\mathrm{d}T} \cdot c_i(x,\mu_m) \right] \quad (25)$$

The over-all solving procedure is as follows:

(1) Assume initial temperature distribution.
(2) Calculate the absorption coefficient a_λ.
(3) Calculate A and B according to expression (13c, d).
(4) Solve equation (14) for $c_i(x,\mu_m)$.
(5) Calculate the radiative intensity and heat flux by expression (21) and (22).
(6) If the energy equation (23) is satisfied, then stop the calculation. Otherwise, adjust the temperature distribution according to expression (24).
(7) Repeat steps (2) to (6) until the energy equation is satisfied. The flow chart is shown in Fig. 3.

3. Results and discussions

As a simple example, CO_2 is chosen to be the absorbing medium. Since for CO_2 the 4.3 μ absorption band is much stronger than others, for simplicity only the 4.3 μ band is considered here. The absorption coefficient is evaluated by

$$a_\lambda = \left(\frac{\rho S}{d}\right) \frac{\sinh(2\pi\gamma/d)}{\cosh(2\pi\gamma/d) - 1} \quad (26)$$

where ρ is density, S/d is mean-line-intensity-to-spacing ratio, and γ/d is line-width-to-spacing ratio for the considered narrow band [17]. The parameters S/d and γ/d incorporate the effects of wavelength, temperature, and pressure and can be found in ref [18].

In order to compare with other methods, two different cases were selected:

case 1: $T_1 = 1500$ K, $T_2 = 400$ K, $P_{CO_2} = 0.2$ atm, $P_{N_2} = 10$ atm, $D = 1$ cm

case 2: $T_1 = 1500$ K, $T_2 = 400$ K, $P_{CO_2} = 1.0$ atm, $P_{N_2} = 0$ atm, $D = 10$ cm

These two cases represent the optically thin and thick problems respectively.

The purpose of this work is to discuss the spectrum discretization, so the angular and spatial discretizations are chosen to provide grid independent results. In angular domain, the S_8 scheme with equal weights for all directions was used. In spacial domain, the first order backwards scheme with 100 grids was chosen. First we solved the optical thin problem, case 1, with $N = 8$, 16 and 32. The convergence of the heat flux and temperature distributions are shown in Table 1 and Fig. 4. Compared with the method in [15], the present method converged quickly. In [15], the same problem was solved in whole spectral range from zero to certain point beyond which the blackbody intensity can be considered zero. Their results were not converged until $N = 256$, but because of the limit of the computer memory and CPU time, results with larger N can not be obtained. Since the absorption band is usually quite small compared to the total radiation range (0.612 to 60 μ in this case), when solving the RTE in the total range, to get a detailed approximation of the properties in so small a band a high-level wavelet function is needed (see Fig. 5). While the present method solves the RTE only in the absorption band, the absorption coefficient can be better approximated with low-level wavelets and less expansion terms, and thus save a great amount of memory and CPU time. If assuming the initial dimensionless temperature distribution $(E - E_2)/(E_1 - E_2)$ (E is the emissive power σT^4) as linear from 1 to 0, the computing procedure on an Ultra-1 took about 400 and 1000 s for $N = 16$ and 32 respectively. The results can serve as the initial conditions for larger N cases. For

Table 1
Convergence of dimensionless heat flux $q_r/\sigma T_2^4$ for case 1

Present method	Bayazitoglu et al. [15]
194.90 ($N = 8$)	193.85 ($N = 64$)
194.96 ($N = 16$)	194.14 ($N = 128$)
194.99 ($N = 32$)	194.55 ($N = 256$)

Table 2
Comparison of dimensionless heat flux $q_r/\sigma T_2^4$

D (cm)	1 (case 1)	10 (case 2)
Transparent gas	196.754	196.754
Wide-band Curtis–Godson method [7]	195.029	189.481
P-1 approximation [17]	194.1	189.2
Wavelet plus P-1 method [14]	194.1961	179.4396
Present work ($M = 8$, $N = 64$)	195.00	190.03

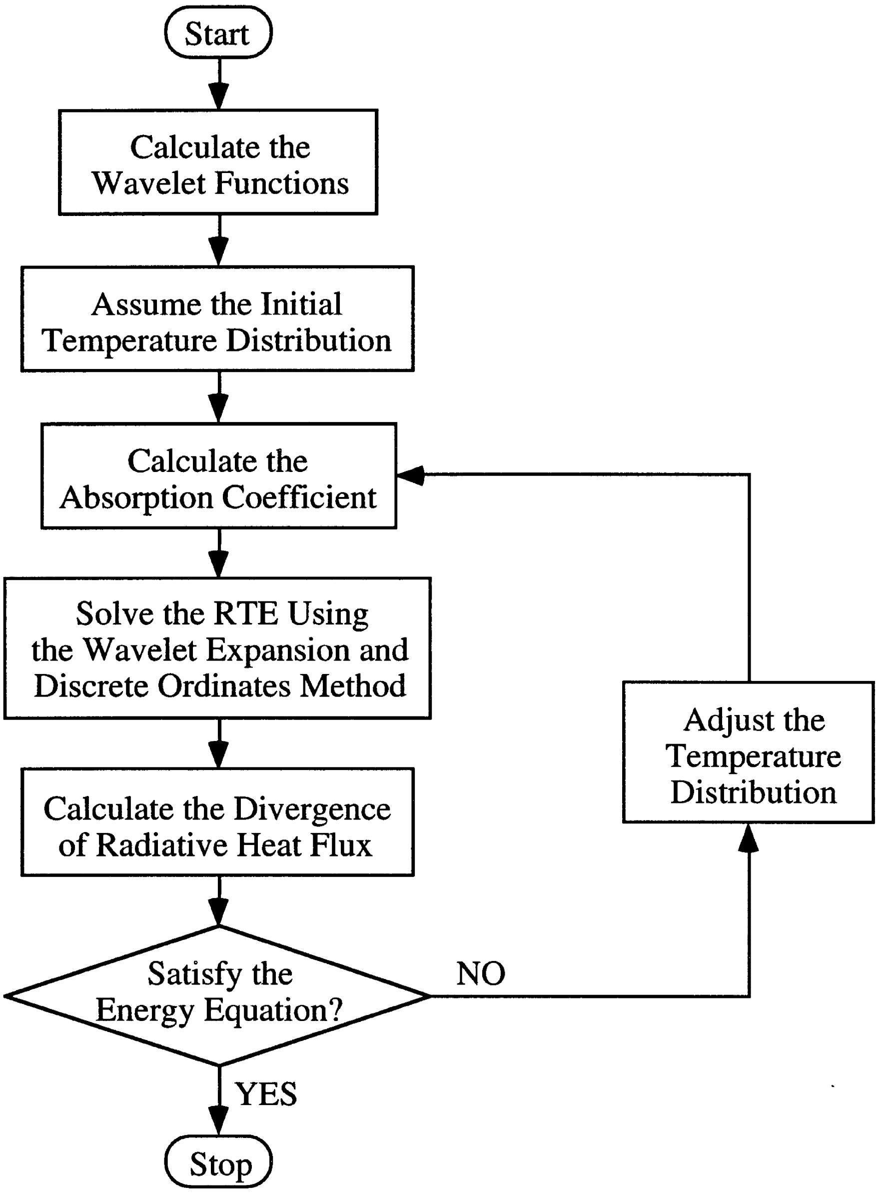

Fig. 3. Flow chart of solving procedure.

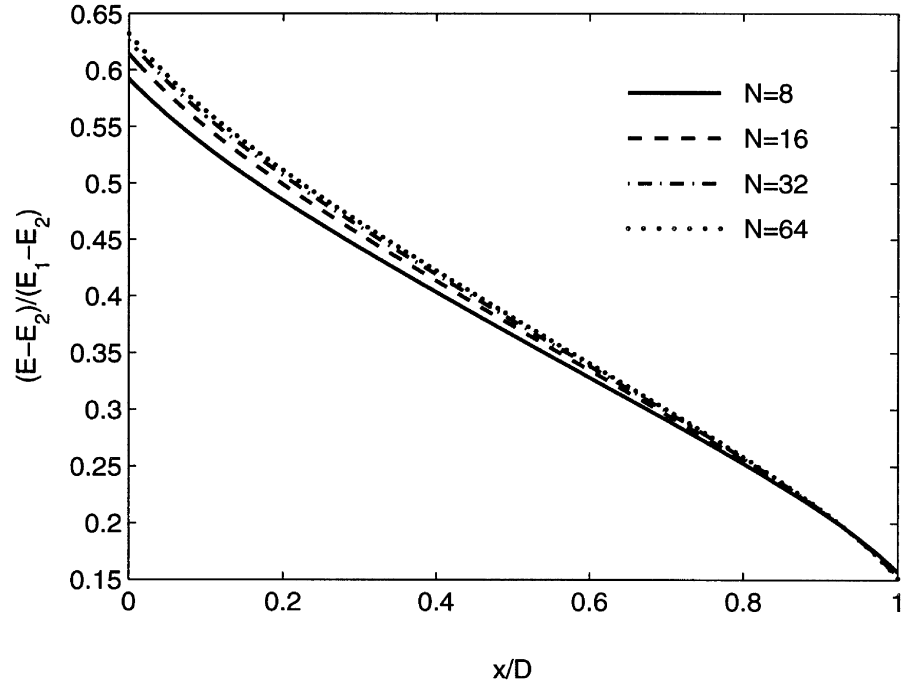

Fig. 4. Nondimensional temperature distribution for case 1.

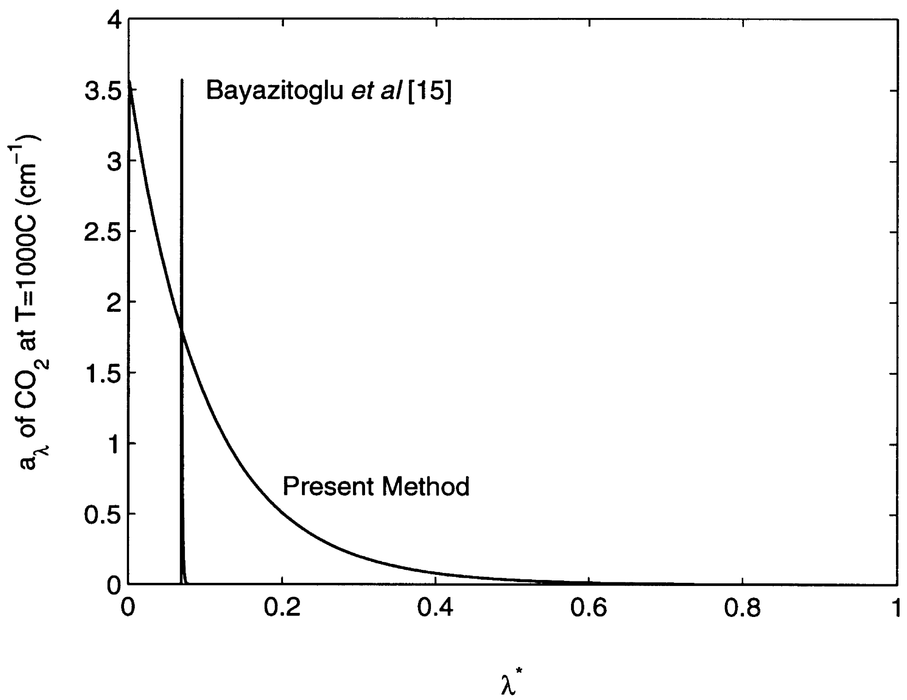

Fig. 5. Absorption coefficient distributions.

the optical thick problem, the computing took a little more time since more iterations were needed.

The comparison of the present work with those of others is shown in Table 2 and Figs 6 and 7. When consulting the references, we found that most methods can give accurate predictions of the heat flux, but few can reach the accurate temperature profiles. The same problem was encountered here. From Table 2, it can be

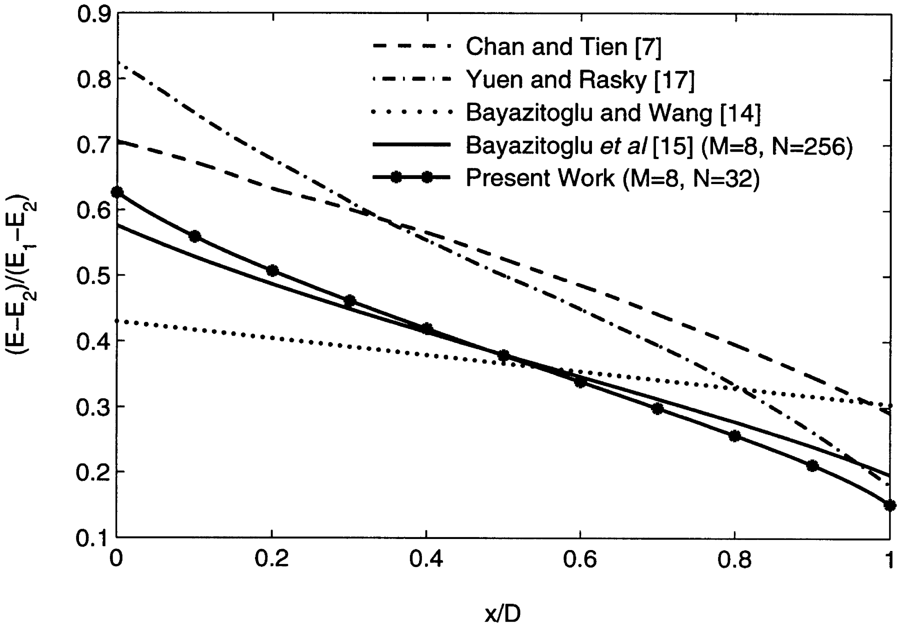

Fig. 6. Nondimensional temperature distributions for case 1.

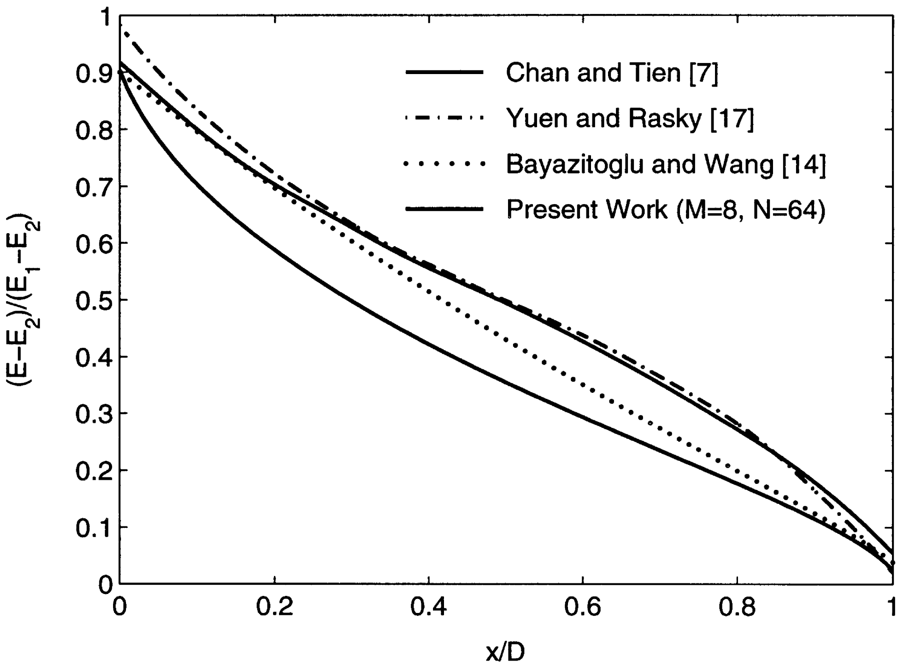

Fig. 7. Nondimensional temperature distributions for case 2.

seen that the differences of heat flux between all the methods for both case 1 and case 2 are less than 0.5% (except for case 2 by wavelet plus P-1 method [14]). However, in Figs 6 and 7, the differences between the temperature distributions are much larger. These differences can be caused by several reasons. Chan and Tien [7] used the total band absorptance of the medium in their analysis, while others used the spectral absorption coefficient as in reference [17]; Chan and Tien [7] assumed the polynomial temperature distribution, Yuen and Rasky [17] assumed the polynomial distribution of zeroth moment in P-1 approximation, which resulted in that the absorption coefficient was evaluated at average temperature, whereas the wavelet plus P-1 method [14] and present work did not make any assumptions. Some similar conclusions from all studies are: the temperature distribution is not antisymmetric with respect to the middle of the two pates; the heat flux decreases with the increase of optical thickness.

The method can be easily extended to multi-dimensional, multi-band problems, although only one-dimensional, one absorption band case was considered in this work. For multi-band problems (without band overlap), the same procedure for solving RTE can be applied to every particular band. For multi-dimensional problems, the resulting coupled partial differential equations can be easily solved by the control volume method (Fiveland [19]).

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